

CENTRALIZERS OF FINITE SUBGROUPS OF THE MAPPING CLASS GROUP

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ABSTRACT. In this paper, we study the action of finite subgroups of the mapping class group of a surface on the curve complex. We prove that if the diameter of the almost fixed point set of a finite subgroup H is big enough, then the centralizer of H is infinite.

1. INTRODUCTION

Let S be an orientable surface of finite type with complexity at least 4, $\text{Mod}(S)$ be the mapping class group of S , $C(S)$ be the curve complex of S and δ be the hyperbolicity constant of $C(S)$. (See Section 4 for the definitions of the above objects, and references.) We prove the following theorem.

Main Theorem. *Let H be a finite subgroup of $\text{Mod}(S)$. Let $C_H = \{\nu \in C(S) : \text{diam}(H \cdot \nu) \leq 6\delta\}$. There exists a constant D , depending only on the topological type of S , such that if $\text{diam}(C_H) \geq D$, then the centralizer of H in $\text{Mod}(S)$ is infinite.*

We call points in C_H *almost fixed points* of H . Note that C_H is never empty. In fact, almost fixed points are very easy to find. Let $\nu \in C(S)$. Then any 1-quasi-centers of the H -orbit of ν are in C_H . (See [4, Lemma III.F.3.3, p.460] for more detail.)

One of the motivations of the Main Theorem is the following: Consider a sequence of homomorphisms $\{f_i\}$ from a finitely generated group G to $\text{Mod}(S)$. This sequence of homomorphisms induce a sequence of actions of G on $C(S)$. Suppose that the translation lengths (with respect to some finite generating set of G) of these actions go to infinity. In this case, these actions of G on $C(S)$ converge to a non-trivial action of G on an \mathbb{R} -tree. The Main Theorem provides some information about this action.

Corollary 1.1. *Let T be the \mathbb{R} -tree obtained as above. Let K be the stabilizer in G of a non-trivial segment in T . Then there exists N , such that any finite subgroup H of $f_i(K)$ has infinite centralizer in $\text{Mod}(S)$ for all $i \geq N$.*

The same phenomenon shows up when one considers the action of a hyperbolic group on its Cayley graph. We include the proof of the Main Theorem for hyperbolic group (Theorem 3.1) in this paper for the following reasons: First, even through experts in geometric group theory might know the proof for hyperbolic groups, as far as the author knows the proof is not in the literature. Second, since the two proofs are similar while the mapping class group case requires many more tools (such as Masur and Minsky's theory of hierarchies) and is more technical, we think that the proof of the hyperbolic group case serves well as a warm-up.

The proofs of both Main Theorems are based on a general fact proved in Section 2. Consider a “nice” finitely generated group G admitting a “nice” action on a infinite metric graph. Lemma 2.1 says if the cardinality of the set of almost fixed points (see Section 2 for definition) of a finite subgroup is big enough, then the centralizer of the finite subgroup is infinite.

In Section 3, we use the hyperbolicity of the Cayley graph of a hyperbolic group to show that having two almost fixed points being far apart implies having a lot of points with small H -orbit. This is Lemma 3.2. Then we show that the action in this case is “nice” in the sense of Lemma 2.1 and main theorem for hyperbolic groups (Theorem 3.1) follows. In Section 4, we introduce the basic definitions we need to state the main theorem. In Section 5, we prove the Main Theorem for the mapping class group. The proof of main theorem for the mapping class group relies heavily on the theory of hierarchies. Readers who are not familiar with the theory of hierarchies should read [13]. In Section 6, we prove Corollary 1.2.

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2. THE KEY LEMMA

Lemma 2.1 is a key fact we need in the proofs of the main theorems, in both the hyperbolic case and the mapping class group case.

In order to state Lemma 2.1, we need to introduce some notation. Consider a finitely generated group G acting properly and cocompactly on a infinite locally finite metric graph K by isometries. Let H be a finite subgroup of G . Let a be a positive integer.

Suppose the cardinalities of finite subgroups of G are bounded above by some number C_0 .

Let $K^{(0)}$ be the set of vertices of K and C_1 be the number of points in $K^{(0)}/G$.

For $p \in K$, let $B(p, a)$ denote the a -neighborhood of p in K and $\text{card}_v(B(p, a))$ be the number of vertices in $B(p, a)$. Let C_2 be an upper bound for $\{\text{card}_v(B(p, a)) : p \in K^{(0)}\}$.

Let $C_3 = \text{Max}\{\text{card}(\text{stab}(p)) : p \in K^{(0)}\}$, where $\text{stab}(p)$ is the stabilizer of p in G .

Lemma 2.1. *Let $P_H = \{p \in K^{(0)} : \text{diam}(H \cdot p) \leq a\}$. Then there exists a constant N , depending only on C_0, C_1, C_2, C_3 , such that if $\text{card}(P_H) \geq N$, the centralizer of H in G is infinite.*

Proof. It suffices to take $N = ((C_0 + 1)(C_3)^{C_0} + 1)C_1(C_2)^{C_0}$. Assume $\text{card}(P_H) \geq N$. We show that in this case the centralizer of H is infinite.

By definition, C_1 is the number of G -orbits in $K^{(0)}$. By the pigeon-hole principle, there are at least

$$r_1 = \frac{N}{C_1}$$

points of P_H in the same orbit. Choose a subset $P = \{p_1, \dots, p_{r_1}\}$ of P_H so that all elements of P are in the same G -orbit. Choose $g_i \in G$ so that $g_i \cdot p_1 = p_i$ for $2 \leq i \leq r_1$. Note that g_i^{-1} induces an isometry from $B(p_i, a)$ to $B(p_1, a)$.

Let $H = \{h_1, \dots, h_d\}$. First, we consider the action of h_1 . For any $p_i \in P$, we have $h_1 \cdot p_i \in B(p_i, a)$ by the definition of P_H . Therefore, $g_i^{-1} \cdot h_1 \cdot p_i \in B(p_1, a)$. Since $\text{card}_v(B(p_1, a)) \leq C_2$, by the pigeonhole principle, there exists $v_1 \in B(p_1, a)$ such that for at least $\frac{r_1}{C_2}$ many i , $g_i^{-1} \cdot h_1 \cdot p_i = v_1$. Let I_1 be the subset of $\{1, \dots, r_1\}$ such that for any $i \in I_1$, we have $g_i^{-1} \cdot h_1 \cdot p_i = v_1$, which is equivalent to $h_1 \cdot p_i = g_i \cdot v_1$.

Now consider h_2 . As above, by the pigeonhole principle, there exists $v_2 \in B(p_1, a)$, and a subset I_2 of I_1 with $\text{card}(I_2) \geq \frac{r_1}{(C_2)^2}$, such that $h_2 \cdot p_i = g_i \cdot v_2$ for all $i \in I_2$.

Repeat this process for all the elements of H , we have

$$h_t \cdot p_i = g_i \cdot v_t$$

for all $1 \leq t \leq d$ and all $i \in I_d$, where $I_d \subset I_{d-1} \subset \dots \subset I_1$ and

$$r_2 = \text{card}(I_d) \geq \frac{r_1}{(C_2)^d}.$$

Fix an element $b \in I_d$. For any $i \in I_d$, we have:

$$\begin{aligned}
h_1 \cdot g_i \cdot g_b^{-1} \cdot p_b &= h_1 \cdot g_i \cdot p_1 \\
&= h_1 \cdot p_i \\
&= g_i \cdot v_1 \\
&= g_i \cdot g_b^{-1} \cdot h_1 \cdot p_b
\end{aligned}$$

Therefore we have:

$$h_1^{-1} \cdot g_b \cdot g_i^{-1} \cdot h_1 \cdot g_i \cdot g_b^{-1} \in \text{stab}(p_b)$$

We know that $\text{card}(\text{stab}(p_b)) \leq C_3$. Now apply the pigeonhole principle again, we know that there exists a subset I_d^1 of I_d with $\text{card}(I_d^1) \geq \frac{r_2-1}{C_3}$, such that for any $i, j \in I_d^1$,

$$h_1^{-1} \cdot g_b \cdot g_i^{-1} \cdot h_1 \cdot g_i \cdot g_b^{-1} = h_1^{-1} \cdot g_b \cdot g_j^{-1} \cdot h_1 \cdot g_j \cdot g_b^{-1},$$

which is equivalent to:

$$g_j \cdot g_i^{-1} \cdot h_1 = h_1 \cdot g_j \cdot g_i^{-1}.$$

Repeat this process for all the elements of H , we get a subset I_d^d of I_d , with $\text{card}(I_d^d) \geq \frac{r_2-1}{(C_3)^d}$, such that for any $i, j \in I_d^d$, any $1 \leq t \leq d$,

$$g_j \cdot g_i^{-1} \cdot h_t = h_t \cdot g_j \cdot g_i^{-1}.$$

Fix $c \in I_d^d$. Then for all $i \in I_d^d$, all $h_t \in H$, we have:

$$g_c \cdot g_i^{-1} \cdot h_t = h_t \cdot g_c \cdot g_i^{-1}$$

Hence $g_c \cdot g_i^{-1}$ centralizes H for all $i \in I_d^d$. Therefore, there are at least $\text{card}(I_d^d)$ elements in the centralizer of H . But since $N = ((C_0 + 1)(C_3)^{C_0} + 1)C_1(C_2)^{C_0}$, we have:

$$r_1 = \frac{N}{C_1} = ((C_0 + 1)(C_3)^{C_0} + 1)(C_2)^{C_0}.$$

Therefore, since $d \leq C_0$, we have:

$$r_2 \geq \frac{r_1}{(C_2)^d} \geq (C_0 + 1)(C_3)^{C_0} + 1.$$

So, again using the fact that $d \leq C_0$, we have:

$$\text{card}(I_d^d) \geq \frac{r_2 - 1}{(C_3)^d} \geq C_0 + 1$$

So there are at least $C_0 + 1$ elements in the centralizer of H , but any finite subgroup of G has cardinality at most C_0 , so the centralizer of H must be infinite. \square

3. MAIN THEOREM AND PROOF - THE HYPERBOLIC GROUP CASE

We use the convention that a δ -hyperbolic space is a geodesic metric space in which all geodesics triangles are δ -thin. (See [4, Definition III.H 1.16, p.408] for more detail.)

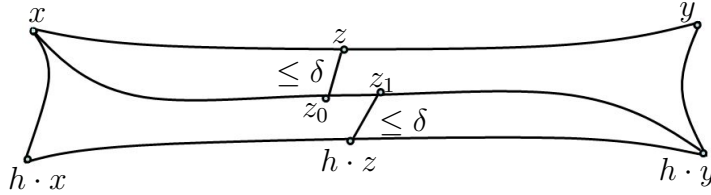
Theorem 3.1. *Let G be a hyperbolic group with $\{g_1, \dots, g_n\}$ as an generating set. Let K_G be the Cayley graph of G with respect to the given generating set. Let δ be the hyperbolicity constant for K_G . Let H be a finite subgroup of G . Let*

$$X_H = \{x \in K_G : \text{diam}(H \cdot x) \leq 6\delta\}.$$

There exists a constant D , depending only on δ and n , such that if $\text{diam}(X_H) \geq D$, then the centralizer of H in G is infinite.

We call $x \in X_H$ *almost fixed points* of H .

Lemma 3.2. *Let $x, y \in X_H$. Suppose $d(x, y) \geq 20\delta$. Let $[x, y]$ be a geodesic in K_G connecting x and y . Then for any vertex $z \in [x, y]$ such that $d(x, z) \geq 6\delta + 1$ and $d(z, y) \geq 6\delta + 1$, we have $\text{diam}(H \cdot z) \leq 8\delta$.*



Proof. It suffices to prove that $d(h \cdot z, z) \leq 8\delta$ for all $h \in H$.

Consider the geodesic triangle with edges:

$$[x, y], [x, h \cdot y], [y, h \cdot y].$$

K_G is δ -hyperbolic, so the triangle satisfies the thin triangle condition. Since $d(z, y) \geq 6\delta + 1$ and $d(y, h \cdot y) \leq 6\delta$, there is a point $z_0 \in [x, h \cdot y]$ such that $d(z, z_0) \leq \delta$ and $d(x, z_0) = d(x, z)$.

Now consider the triangle with edges

$$[x, h \cdot x], [x, h \cdot y], [h \cdot x, h \cdot y] = h \cdot [x, y].$$

As above, since $d(h \cdot z, h \cdot x) = d(x, z) \geq 6\delta + 1$ and $d(x, h \cdot x) \leq 6\delta$, there is a point $z_1 \in [x, h \cdot y]$ such that $d(h \cdot z, z_1) \leq \delta$ and $d(h \cdot y, z_1) =$

$d(h \cdot y, h \cdot z)$. So we have:

$$\begin{aligned}
d(z_0, z_1) &= |d(x, z_0) + d(h \cdot y, z_1) - d(x, h \cdot y)| \\
&= |d(x, z) + d(h \cdot y, h \cdot z) - d(x, h \cdot y)| \\
&= |d(h \cdot x, h \cdot z) + d(h \cdot y, h \cdot z) - d(x, h \cdot y)| \\
&= |d(h \cdot x, h \cdot y) - d(x, h \cdot y)| \\
&\leq 6\delta.
\end{aligned}$$

Now we know: $d(h \cdot z, z) \leq d(z, z_0) + d(h \cdot z, z_1) + d(z_0, z_1) \leq \delta + \delta + 6\delta = 8\delta$. \square

Apply Lemma 2.1 to the action of G on K_G , we get the following lemma.

Lemma 3.3. *Let H and G be as in Theorem 3.1. Let $P_H = \{x \in K_G : \text{diam}(H \cdot x) \leq 8\delta\}$. There exists a constant N , depending only on δ and n , such that if $\text{card}(P_H) \geq N$, then the centralizer of H in G is infinite.*

Proof. In order to apply Lemma 2.1, it suffices to show that in the current situation, C_0, C_1, C_2, C_3 are finite and they depend only on δ and n .

By [4, Theorem III.Γ.3.2, p.459], there exists an upper bound, depending only on δ and n , for the cardinality of finite subgroups of G . So C_0 is finite and depends only on δ and n . We have $C_1 = 1$ since K_G/G has only one vertex. Also C_2 is finite and depends only on δ and n by the definition of Cayley graph. Finally, $C_3 = 1$ since the action is free. \square

Proof of Theorem 3.1. Let $D = N + 12\delta + 4$, where N is the constant given by the previous lemma. Then D depends only on δ and n . Let $x, y \in X_H$ such that $d(x, y) \geq D$. Let $[x, y]$ be a geodesic connecting x, y . Let $B = \{z \in [x, y] \mid d(z, x) \geq 6\delta + 1, d(z, y) \geq 6\delta + 1\}$. Then $\text{card}(B) \geq N$ and $B \subset P_H$, where P_H is as in the statement of Lemma 3.3. So $\text{card}(P_H) \geq N$. Therefore, by Lemma 3.3, the centralizer of H in G is infinite. \square

4. Mod(S): BASIC DEFINITIONS

Let $S = S_{\gamma, p}$ be an orientable surface of finite type, with genus γ and p punctures. The only surfaces with boundary we consider will be subsurfaces of S . The *complexity* of S is measured by $\xi(S) = 3\gamma(S) + p(S)$. In this paper, we only consider surfaces with $\xi \geq 4$. The only exception is the annulus, which will only appear as a subsurface of S .

The *Mapping Class Group* of S , denoted by $\text{Mod}(S)$, is the group of orientation-preserving homeomorphisms of S modulo isotopy.

A *curve* on S will always mean the isotopy class of a simple closed curve, which is not null-homotopic or homotopic into a puncture.

For surface S with $\xi \geq 5$, the *graph of curves* $C(S)$ consists of a vertex for every curve, with edges joining pairs of distinct curves that have disjoint representatives on S . The graph of curves is the 1-skeleton of the curve complex introduced by Harvey.

When $\xi = 4$, the surface S is either a once-punctured torus $S_{1,1}$ or four times punctured sphere $S_{0,4}$. We have an alternate definition for the graph of curves $C(S)$: Vertices are still curves. Edges are given by pairs of distinct curves that have representatives that intersect once (for $S_{1,1}$) or twice (for $S_{0,4}$).

By assigning length 1 to each edge we make $C(S)$ into a metric graph. We use d_S to denote this metric. Masur and Minsky prove the following theorem ([12, Theorem 1.1]).

Theorem 4.1. *$C(S)$ is an δ -hyperbolic metric space, where δ depends on S . Except when S is a sphere with 3 or fewer punctures, $C(S)$ has infinite diameter.*

Since elements in $\text{Mod}(S)$ preserve disjointness of curves, $\text{Mod}(S)$ acts on $C(S)$ by isometry. This action is cocompact since there are only finitely many curves on S up to homeomorphisms, but it is far from proper.

Convention 4.2. *For the rest of the paper, by an element $x \in C(S)$ we always mean a vertex of $C(S)$ and similarly for a subset of $C(S)$.*

5. MAIN THEOREM AND PROOF

In this section we prove the Main Theorem for $\text{Mod}(S)$. First, recall the statement.

Theorem 5.1 (Main). *Let H be a finite subgroup of $\text{Mod}(S)$. Let*

$$C_H = \{\nu \in C(S) : \text{diam}(H \cdot \nu) \leq 6\delta\}.$$

There exists a constant D , depending only on the topological type of S , such that if $\text{diam}(C_H) \geq D$, then the centralizer of H in $\text{Mod}(S)$ is infinite.

Proof. Just as in the hyperbolic groups case, we first show that having two almost fixed points being far apart implies having a lot of points with small H -orbit. The idea of the following lemma is the same as Lemma 3.2.

Lemma 5.2. *Let $\nu_0, \nu_1 \in C_H$. Suppose $d(\nu_0, \nu_1) \geq 20\delta$. Let $[\nu_0, \nu_1]$ be a geodesic in $C(S)$ connecting ν_0 and ν_1 . Then for any vertex $b \in [\nu_0, \nu_1]$ such that $d_S(\nu_0, b) \geq 6\delta + 1$ and $d_S(b, \nu_1) \geq 6\delta + 1$, we have $\text{diam}_S(H \cdot b) \leq 8\delta$.*

If we can apply Lemma 2.1 to prove a similar result as Lemma 3.3 for the action of $\text{Mod}(S)$ on $C(S)$, the Main Theorem will follow. But one immediately sees that such result can not be proved in the same way for two reasons: $C(S)$ is locally infinite and action of $\text{Mod}(S)$ on $C(S)$ has infinite vertex-stabilizers. However, we can prove a similar result for a “nicer” action of $\text{Mod}(S)$ on a locally finite graph.

Let $\mathcal{M}(S)$ be the graph of complete clean markings of the surface S as defined by Masur and Minsky in [13, section 7.1]. We use $d_{\mathcal{M}}$ to denote the metric on $\mathcal{M}(S)$. Recall that $\mathcal{M}(S)$ is locally finite and admits an proper and cocompact action by $\text{Mod}(S)$ by isometries. Apply Lemma 2.1 to the action of $\text{Mod}(S)$ on $\mathcal{M}(S)$. We get the following lemma.

Lemma 5.3. *Let a be any positive integer. Let H be a finite subgroup of $\text{Mod}(S)$. Let $P_H^a = \{P \in \mathcal{M}(S) : \text{diam}(H \cdot P) \leq a\}$. There exists a constant N , depending only on S and a , such that if $\text{card}(P_H^a) \geq N$, the centralizer of H is infinite.*

Proof. In order to apply Lemma 2.1, it suffices to show that in the current situation, C_0, C_1, C_2, C_3 are finite and they depend only on S and a .

By Nielsen Realization Theorem (See [17] for a proof for the case of puncture surfaces) every finite subgroup of $\text{Mod}(S)$ can be realized as a subgroup of the isometry group of the surface with some hyperbolic structure. By Hurwitz’s automorphisms theorem, the size of the isometry group of a punctured hyperbolic surface is bounded above. (The bound is $84(g - 1)$ when $g \geq 2$. When $g \leq 1$, a similar argument as in [8, Section 7.2] gives an upper bound for the size of the isometry group.) Hence the orders of finite subgroups of $\text{Mod}(S)$ are bounded above by a constant which depends only on the topological type of S . So C_0 is finite and depends only on S . By the construction of $\mathcal{M}(S)$, both C_1 and C_3 are finite and depend only on S . For the same reason, C_2 is finite and depends only on S and a . \square

Lemma 5.2 and Lemma 5.3 together do not give the result we want since they are about actions of $\text{Mod}(S)$ on different metric spaces. In order to connect these two actions, we use Masur and Minsky’s theory of hierarchies.

Let ν_0, ν_1 be C_H . Let μ_0, μ_1 be markings ([13, section 2.5]) such that $\nu_0 \in \text{base}(\mu_0)$, $\nu_1 \in \text{base}(\mu_1)$. Let $\mathcal{H} = [\mu_0, \mu_1]$ be a hierarchy ([13, Definition 4.4]) with initial marking μ_0 , terminal marking μ_1 and with the main geodesic connecting ν_0, ν_1 . For $h \in H$, Let \mathcal{H}_h be the h translate of \mathcal{H} .

Let B be the set of vertices in $[\nu_0, \nu_H]$, the main geodesic of \mathcal{H} , such that $d_S(\nu_0, b) \geq 14\delta + 5$ and $d_S(b, \nu_1) \geq 14\delta + 5$. For any $b \in B$, $h \in H$, let μ_b be a marking compatible with a slice ([13, section 5]) of \mathcal{H} at b . Then $h \cdot \mu_b$ is a marking compatible with a slice of \mathcal{H}_h at $h \cdot b$. Let $\mathcal{H}_b^h = [\mu_b, h \cdot \mu_b]$ be a hierarchy connecting μ_b and $h \cdot \mu_b$.

Lemma 5.4. \mathcal{H}_b^h is (K, M') -pseudo-parallel ([13, Definition 6.5]) to \mathcal{H} where K and M' depend only on S .

Proof. By Lemma 5.2, the main geodesic $[\nu_0, \nu_H]$ of \mathcal{H} and the main geodesic $h \cdot [\nu_0, \nu_H]$ of \mathcal{H}_h are $(8\delta + 2, 2\delta + 1)$ -parallel ([13, Definition 6.4]) at b and $h \cdot b$ for all $b \in B$ and $h \in H$. Now apply [13, Lemma 6.7]. \square

Before we can define the constant D in the Main Theorem, we need the following lemma.

Lemma 5.5. Let \mathcal{H} be a hierarchy. Let c be any positive number. Suppose that the lengths of all the geodesics in \mathcal{H} are less than c . Then the distance between the initial marking and the terminal marking of \mathcal{H} in $\mathcal{M}(S)$ is less than d , where d is a number depending only on c and the topological type of S .

Proof. Apply [13, Theorem 6.12] with $M = c$. \square

Let M be the constant in [13, Theorem 3.1]. Let M_1, M_2 be the constants in [13, Lemma 6.2]. Let K and M' be the constants in Lemma 5.4. Let $e = 2M + 8M_1 + M_2 + 2K + M'$. Let d be the constant given by Lemma 5.5 with the above $c = e + 2M_1$. Let N be the constant given by Lemma 5.3 with $a = d$. Let $D = N + 12\delta + 10$. We will show this is the constant D we want. Note that D depends only on the topological type of S .

The rest of the proof is devoted to showing that the centralizer of H is infinite provided that $d_S(\nu_0, \nu_1) \geq D$.

The proof will break into 2 cases: If the length of the hierarchies \mathcal{H}_b^h are bounded for all $b \in B$, $h \in H$, then the distance between μ_b and $h \cdot \mu_b$ in $\mathcal{M}(S)$ are bounded. In this case, we have enough almost-fixed points in $\mathcal{M}(S)$ and we can apply Lemma 5.3 to conclude that the centralizer of H in $\text{Mod}(S)$ is infinite. On the other hand, if

there is a “long” hierarchy H_b^h , we are able to use an argument in Jing Tao’s thesis [16] to show that there exists a subsurface Y of S such that elements of H either preserve Y or take Y completely off itself. Using this fact we construct an infinite order element of $\text{Mod}(S)$ which centralizes H .

Case 1: For any $b \in B$, $h \in H$ and any subsurface Y of S supporting a geodesic of H_b^h , $d_Y(\mu_b, h \cdot \mu_b) \leq e$. (See [13, section 2.3] for the definition of d_Y .)

Claim 1. *In Case 1, $d_{\mathcal{M}}(\mu_b, h \cdot \mu_b) \leq d$ for all $b \in B$, $h \in H$, where d is one of the numbers we used to define D .*

Proof. By [13, Lemma 6.2], the geodesic in Y has length at most $e + 2M_1$. Now the claim follows from Lemma 5.5 and the definition of d . \square

Note that Claim 1 says that for any $b \in B$, μ_b is in P_H^d . Since $d_S(\nu_0, \nu_1) \geq D$, we have $|P_H^d| \geq |B| \geq D - 12\delta - 8 \geq N$. By Lemma 5.3 and the definition of N , the centralizer of H is infinite and the proof is complete in Case 1.

Case 2: There exists $b_l \in B$, $h_l \in H$, and a subsurface Y of S which supports a geodesic of $\mathcal{H}_{b_l}^{h_l}$, such that $d_Y(\mu_{b_l}, h_l \cdot \mu_{b_l}) \geq e$.

Lemma 5.6. *In Case 2, $d_Y(\mu_0, \mu_1) \geq 2M + 4M_1 + M_2$.*

Proof. Since we are in Case 2 we have $d_Y(\mu_{b_l}, h_l \cdot \mu_{b_l}) \geq e \geq M_2$. So by [13, Lemma 6.2] Y supports a geodesic of $\mathcal{H}_{b_l}^{h_l}$ of length at least $e - 2M_1 = 2M + 6M_1 + M_2 + 2K + M'$. In particular, this geodesic has length bigger than M' . By Lemma 5.4, $\mathcal{H}_{b_l}^{h_l}$ is (K, M') -pseudo-parallel to \mathcal{H} . So Y also supports a geodesic of \mathcal{H} , whose length is at least $2M + 6M_1 + M_2 + 2K + M' - 2K = 2M + 6M_1 + M_2 + M'$. Now apply [13, Lemma 6.2] again, we know that $d_Y(\mu_0, \mu_1) \geq 2M + 6M_1 + M_2 + M' - 2M_1 \geq 2M + 4M_1 + M_2$ as we claim. \square

Lemma 5.7. *Let $b \in B$, $h \in H$. Suppose Y supports a geodesic of \mathcal{H}_b^h . Then $d_Y(\mu_0, h \cdot \mu_0) \leq M$ and $d_Y(\mu_1, h \cdot \mu_1) \leq M$.*

Proof. Let $[\nu_b, h \cdot \nu_b]$ be the main geodesic in \mathcal{H}_b^h . Since Y supports a geodesic in \mathcal{H}_b^h , Y must be forward subordinate (See [13, section 4.1] for definition.) to $[\nu_b, h \cdot \nu_b]$ at some vertex ν . Let l be any boundary component of Y . Then $d_S(l, \nu) = 1$. Since $\nu_0 \in C_H$, we have $d_S(\nu_0, h \cdot \nu_0) \leq 6\delta$. Let $[\nu_0, h \cdot \nu_0]$ be a geodesic connecting $\nu_0, h \cdot \nu_0$. Let ν_i be a

point on $[\nu_0, h \cdot \nu_0]$. By the triangle inequality,

$$\begin{aligned} d_S(\nu, \nu_i) &\geq d_S(\nu_0, \nu_b) - d_S(\nu, \nu_b) - d_S(\nu_i, \nu_0) \\ &\geq d_S(\nu_0, \nu_b) - d_S(\nu_b, h \cdot \nu_b) - d_S(\nu_0, h \cdot \nu_0) \\ &\geq (14\delta + 5) - (8\delta + 2) - 6\delta \\ &= 3. \end{aligned}$$

Then $d_S(l, \nu_i) \geq d_S(\nu, \nu_i) - d_S(l, \nu) \geq 3 - 1 = 2$. Therefore ν_i intersects l . As a result, ν_i intersects Y . And this is true for all $\nu \in [\nu_0, h' \cdot \nu_0]$. By [13, Theorem 3.1], $d_Y(h \cdot \nu_0, \nu_0) \leq M$. An exact same argument shows $d_Y(\mu_1, h \cdot \mu_1) \leq M$. \square

We prove the following key lemma for Case 2 using an argument in [16, Lemma 3.3.4].

Lemma 5.8. *In Case 2, for any $h \in H$, either $h(Y) = Y$ or $h(Y)$ and Y are disjoint.*

Proof. Let $h \in H$. Applying Lemma 5.6 and Lemma 5.7, we have

$$\begin{aligned} d_{h^{-1}(Y)}(\mu_0, \mu_1) &= d_Y(h \cdot \mu_0, h \cdot \mu_1) \\ &\geq d_Y(\mu_0, \mu_1) - d_Y(\mu_0, h \cdot \mu_0) - d_Y(\mu_1, h \cdot \mu_1) \\ &\geq 2M + 4M_1 + M_2 - M - M \\ &= 4M_1 + M_2 \\ &\geq M_2. \end{aligned}$$

So by [13, Lemma 6.2], $h^{-1}(Y)$ is also a domain in \mathcal{H} . Suppose $h^{-1}(Y) \neq Y$. Then since $h^{-1}(Y)$ and Y have the same complexity, they are either disjoint from each other or they interlock (i.e. intersect but do not contain each other).

Suppose $h^{-1}(Y)$ and Y are not disjoint. Then by [13, Lemma 4.18], $h^{-1}(Y)$ and Y are time-ordered ([13, Definition 4.16]).

First suppose $Y \prec_t h^{-1}(Y)$ (Here \prec_t is the notation for time order). As in the proof of [13, Lemma 6.11], there exist a slice in \mathcal{H} so that its associated compatible marking ν satisfies

$$d_Y(\nu, \mu_1) \leq M_1 \quad \text{and} \quad d_{h^{-1}(Y)}(\nu, \mu_0) \leq M_1.$$

Then since $d_{h^{-1}(Y)}(\nu, \mu_0) = d_Y(h \cdot \mu_0, h \cdot \nu)$, we have

$$\begin{aligned} d_Y(\mu_0, h \cdot \nu) &\leq d_Y(\mu_0, h \cdot \mu_0) + d_Y(h \cdot \mu_0, h \cdot \nu) \\ &\leq M + M_1. \end{aligned}$$

By Lemma 5.6, we have

$$\begin{aligned} d_Y(\mu_1, h \cdot \nu) &\geq d_Y(\mu_0, \mu_1) - d_Y(\mu_0, h \cdot \nu) \\ &\geq 2M + 4M_1 + M_2 - (M + M_1) \geq 2M_1. \end{aligned}$$

Therefore, by [6, Lemma 1], we have

$$d_{h^{-1}(Y)}(\mu_0, h \cdot \nu) \leq 2M_1.$$

Hence we get

$$\begin{aligned} d_Y(\mu_0, h^2 \cdot \nu) &\leq d_Y(\mu_0, h \cdot \mu_0) + d_Y(h \cdot \mu_0, h^2 \cdot \nu) \\ &\leq M + d_{h^{-1}(Y)}(\mu_0, h \cdot \nu) \\ &\leq M + 2M_1. \end{aligned}$$

Then Then by Lemma 5.6, we have

$$\begin{aligned} d_Y(\mu_1, h^2 \cdot \nu) &\geq d_Y(\mu_0, \mu_1) - d_Y(\mu_0, h^2 \cdot \nu) \\ &\geq 2M + 4M_1 + M_2 - (M + 2M_1) \geq 2M_1. \end{aligned}$$

Again by [6, Lemma 1], we have

$$d_{h^{-1}(Y)}(\mu_0, h^2 \cdot \nu) \leq 2M_1.$$

Iterating this argument, we get

$$\begin{aligned} d_Y(\mu_0, h^i \cdot \nu) &\leq d_Y(\mu_0, h \cdot \mu_0) + d_Y(h \cdot \mu_0, h^i \cdot \nu) \\ &\leq M + d_{h^{-1}(Y)}(\mu_0, h^{i-1} \cdot \nu) \\ &\leq M + 2M_1. \end{aligned}$$

Since this is true for all $i \geq 0$ and h has finite order, we have

$$d_Y(\mu_0, \nu) \leq M + 2M_1.$$

Hence, we get

$$\begin{aligned} d_Y(\mu_0, \mu_1) &\leq d_Y(\mu_0, \nu) + d_Y(\nu, \mu_1) \\ &\leq M + 2M_1 + M_1 \\ &\leq M + 3M_1 \end{aligned}$$

contradicting Lemma 5.6.

In the same way, we can show that $h^{-1}(Y) \prec_t Y$ cannot happen either. So $h^{-1}(Y)$ and Y are not time ordered and hence are disjoint. Therefore, $h(Y)$ and Y are disjoint provided that $h(Y) \neq Y$ as required. \square

Let A be the set of boundary components of Y and all the H -translates of Y . By Lemma 5.8, A is a set of pairwise disjoint curves. Let $T = \prod_{[\alpha] \in A} D_{[\alpha]}$, where $D_{[\alpha]}$ is the right Dehn twist around α .

Lemma 5.9. *For any $h \in H$, $h \cdot T = T \cdot h$.*

Proof. The idea of the proof is as follow: For any $h \in H$, we pick a representative $h_S \in \text{Homeo}^+(S)$ of h and construct $T_h \in \text{Homeo}^+(S)$ such that $h_S \cdot T_h = T_h \cdot h_S$. So they also commute in $\text{Mod}(S)$. Then we note that for all $h \in H$, $T_h \simeq T$. Therefore $T = T_h$ in $\text{Mod}(S)$.

For $h \in H$, h permutes the elements of A . Let

$$([\alpha_1^1], [\alpha_1^2], \dots, [\alpha_1^{j_1}]), \dots, ([\alpha_n^1], [\alpha_n^2], \dots, [\alpha_n^{j_n}])$$

be the decomposition of A into h -cycles. So we have $h \cdot [\alpha_i^j] = [\alpha_i^{j+1}]$ and $h \cdot [\alpha_i^{j_i}] = [\alpha_i^1]$, for $1 \leq i \leq n$.

For each $[\alpha] \in A$, pick a simple representative α such that representatives of different elements of A are disjoint. Pick a neighborhood $N(\alpha)$ for each α such that neighborhoods of different representatives are disjoint. It is easy to see that we can pick a representative $h_S \in \text{Homeo}^+(S)$ of h such that the following are true for all $1 \leq i \leq n$:

- (1) h_S takes $N(\alpha_i^j)$ to $N(\alpha_i^{j+1})$ by homeomorphism for $j \leq j_i - 1$.
- (2) h_S takes $N(\alpha_i^{j_i})$ to $N(\alpha_i^1)$ by homeomorphism.
- (3) $(h_S)^{j_i}$ is the identity map on $N(\alpha_i^1)$ if $(h)^{j_i}$ preserves the two sides of $[\alpha_i^1]$.

(4) $(h_S)^{j_i}$ is “ π -rotation” on $N(\alpha_i^1)$ if $(h)^{j_i}$ flips the two sides of $[\alpha_i^1]$. Here the “ π -rotation” map is an order 2 orientation preserving map which flips the two boundary components of $N(\alpha_i^1)$.

Next, we define T_h . Let T_h be the identity map on $S - \bigcup_{[\alpha] \in A} N(\alpha)$. For all $1 \leq i \leq n$, let T_h be a right Dehn Twist $T_{\alpha_i^1}$ on $N(\alpha_i^1)$. For $2 \leq j \leq j_i$, let T_h be $T_{\alpha_i^j} = (h_S)^{j-1} \cdot T_{\alpha_i^1} \cdot (h_S)^{1-j}$ on $N(\alpha_i^j)$.

On $S - \bigcup_{[\alpha] \in A} N(\alpha)$, T_h and h_S commute in $\text{Mod}(S)$ since they commute in $\text{Homeo}^+(S)$ as T_h is the identity.

Suppose $1 \leq j \leq j_i - 1$. On $N(\alpha_i^j)$ we have

$$h_S \cdot T_h = h_S \cdot (h_S)^{j-1} \cdot T_{\alpha_i^1} \cdot (h_S)^{1-j} = (h_S)^j \cdot T_{\alpha_i^1} \cdot (h_S)^{1-j}$$

and

$$T_h \cdot h_S = (h_S)^j \cdot T_{\alpha_i^1} \cdot (h_S)^{-j} \cdot h_S = (h_S)^j \cdot T_{\alpha_i^1} \cdot (h_S)^{1-j}.$$

So T_h and h_S also commute in $\text{Homeo}^+(S)$ hence in $\text{Mod}(S)$.

On $N(\alpha_i^{j_i})$, we have

$$h_S \cdot T_h = h_S \cdot (h_S)^{j_i-1} \cdot T_{\alpha_i^1} \cdot (h_S)^{1-j_i} = (h_S)^{j_i} \cdot T_{\alpha_i^1} \cdot (h_S)^{1-j_i}$$

and

$$T_h \cdot h_S = T_{\alpha_i^1} \cdot h_S.$$

If $(h_S)^{j_i}$ is the identity, then $(h_S)^{1-j_i} = h_S$. Again we see that T_h and h_S commute in $\text{Homeo}^+(S)$ hence in $\text{Mod}(S)$.

If $(h_S)^{j_i}$ is the “ π -rotation” f , then $f \cdot (h_S)^{1-j_i} = h_S$. Therefore we have

$$h_S \cdot T_h = (h_S)^{j_i} \cdot T_{\alpha_i^1} \cdot (h_S)^{1-j_i} = f \cdot T_{\alpha_i^1} \cdot (h_S)^{1-j_i}$$

and

$$T_h \cdot h_S = T_{\alpha_i^1} \cdot h_S = T_{\alpha_i^1} \cdot f \cdot (h_S)^{1-j_i}.$$

One can easily check that $f \cdot T_{\alpha_i^1} = T_{\alpha_i^1} \cdot f$ in $\text{Mod}(S)$. So T_h and h_S commute in $\text{Mod}(S)$.

Finally, we note that T_h projects to T in $\text{Mod}(S)$ and the proof of the lemma is complete. \square

The above lemma completes the proof in Case 2 since T has infinite order. Therefore the proof of Theorem 5.1 is complete. \square

6. APPLICATION

In this section we prove Corollary 1.2.

Let G be a finitely generated group with a generating set $\{g_1, \dots, g_n\}$. Let $\{f_i\}$ be a sequence of homomorphisms from G to $\text{Mod}(S)$. The f_i induce a sequence of actions ρ_i of G on $C(S)$, where

$$\rho_i(g)(\nu) = f_i(g) \cdot \nu.$$

Let

$$d_i = \inf_{\nu \in C(S)} (\max_{1 \leq t \leq n} d_S(\nu, f_i(g_t) \cdot \nu)).$$

Suppose d_i goes to infinity as i goes to infinity. Then ρ_i subconverges to a non-trivial action ρ of G on an \mathbb{R} -tree T in the sense of Bestvina-Paulin. Replace ρ_i by a convergent subsequence, which we still denote by ρ_i .

Remark 6.1. *In Paulin’s original construction for hyperbolic groups, d_i goes to infinity as long as f_i are non-conjugate. This is not true for $\text{Mod}(S)$.*

Corollary 6.2. *Let T be the \mathbb{R} -tree obtain as above. Let K be the stabilizer in G of a non-trivial segment in T . There exists N , such that any finite subgroup H of $f_i(K)$ has infinite centralizer in $\text{Mod}(S)$ for all $i \geq N$.*

Proof. Let $[x, y]$ be the non-trivial segment in T stabilized by K . Let $l = d_T(x, y)$ and $\epsilon \leq \frac{1}{10}l$. By the construction of T , for i large enough there exists $x_i, y_i \in C(S)$ such that for all $h \in K$ we have:

$$\left| \frac{1}{d_i} d_S(x_i, y_i) - d_T(x, y) \right| \leq \epsilon;$$

$$\begin{aligned} \left| \frac{1}{d_i} d_S(x_i, f_i(h) \cdot x_i) - d_T(x, \rho(h)x) \right| &\leq \epsilon; \\ \left| \frac{1}{d_i} d_S(y_i, f_i(h) \cdot y_i) - d_T(y, \rho(h)y) \right| &\leq \epsilon. \end{aligned}$$

(See [3, Proposition 3.6] for more detail.) Since $l = d_T(x, y)$ and h fixes $[x, y]$, we have:

$$d_S(x_i, y_i) \geq d_i(l - \epsilon);$$

$$d_S(x_i, f_i(h) \cdot x_i) \leq d_i \epsilon;$$

$$d_S(y_i, f_i(h) \cdot y_i) \leq d_i \epsilon.$$

Therefore the $f_i(K)$ -orbit of x_i has bounded diameter. Let C_{x_i} be a 1-quasi-center (See [4, Lemma III.Γ.3.3, p.460] for the definition) of the $f_i(K)$ -orbit of x_i . Then all the $f_i(K)$ -translates C_{x_i} are also 1-quasi-center of the $f_i(K)$ -orbit of x_i . Therefore by [4, Lemma III.Γ.3.3, p.460],

$$d_S(C_{x_i}, f_i(h) \cdot C_{x_i}) \leq 4\delta + 2 \leq 6\delta.$$

Similarly, we have

$$d_S(C_{y_i}, f_i(h) \cdot C_{y_i}) \leq 4\delta + 2 \leq 6\delta.$$

So x_i, y_i are in $C_{f_i(K)}$, which is defined in Theorem 5.1.

By the definition of quasi-center, we have

$$d_S(C_{x_i}, x_i) \leq \text{diam}(f_i(K) \cdot x_i) \leq d_i \epsilon.$$

$$d_S(C_{y_i}, y_i) \leq \text{diam}(f_i(K) \cdot y_i) \leq d_i \epsilon.$$

and so

$$d_S(C_{x_i}, C_{y_i}) \geq d_i(l - \epsilon) - d_i \epsilon - d_i \epsilon \geq d_i(l - 3\epsilon).$$

Therefore when i is large enough

$$d_S(C_{x_i}, C_{y_i}) \geq D,$$

where D is the constant in Theorem 5.1. Now apply Theorem 5.1 to a finite subgroup H of $f_i(K)$, we know that H has infinite centralizer in $\text{Mod}(S)$. \square

Suppose G splits over a finite segment stabilizer C . ($G = A *_C B$ if G splits as an amalgamated free product). Then Corollary 6.1 allows one to construct homomorphisms from G to $\text{Mod}(S)$ of the following form: $\varphi_i(a) = f_i(a)$ for $a \in A$ and $\varphi_i(b) = z^{-1} f_i(b) z$ for $b \in B$ where z is an element of $\text{Mod}(S)$ which centralizes $f_i(C)$. We think that this type of

homomorphisms might be useful when one tries to use the “shortening argument” (See [1], [10], [14], [15]) to study $\text{Hom}(G, \text{Mod}(S))$.

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